Solving constant-size geometric problems using Computer algebra

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- **Theme:** Nonlinear Computational Geometry
- **Lab:**
- **Teams:** SALSA (http://fgbrs.lip6.fr/salsa/) & VEGAS (http://vegas.loria.fr)
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General presentations

The area of *Nonlinear computational geometry* studies the design of algorithmic solutions to geometric problems that involve nonlinear objects. It is at the interface between *Computational Geometry*, which deals with the design of algorithmic solutions to geometric problems, and *Computer Algebra*, one of whose topic is the resolution of polynomial systems.

The goal of this internship is to tackle constant-size geometric problems. Specifically, we focus on problems originating from *geometric transversal theory*.

Let C be a collection of 4 balls in $\mathbb{R}^3$. Denote by $T(C)$ (resp. $I(C)$) the set of lines that are tangent to (resp. intersect) every member of C. The structure of these two sets is still not well understood, and the following are examples of questions that have remained open for several years:

- How many points can $T(C)$ have when the balls are pairwise disjoint and have nonaligned centers? The answer is known to be between 8 and 12.
- What is the maximum number of connected components that $I(C)$ can have when the balls have radius 1 and are pairwise disjoint? The answer is known to be either 2 or 3.

These problems can easily be recast into questions of real algebraic geometry. For instance, using the Plücker coordinates of lines, the set of lines tangent to the ball with center (a, b, c) and radius r in $\mathbb{R}^3$ is the intersection, in $\mathbb{P}^5$, of the Plücker quadric

$$x_1x_4 + x_2x_5 + x_3x_6 = 0$$

and the quadratic hypersurface

$$(x_4 + x_2c - x_3b)^2 + (x_5 + x_3a - x_1c)^2 + (x_6 - x_2a + x_1b)^2 - r^2(x_1^2 + x_2^2 + x_3^2) = 0.$$

Plugging the naive algebraic formulation in existing computer algebra systems fails. The goal of this internship is to combine a better understanding of the geometric problem and of the strengths and limits of the tools developped in the area of effective real algebraic geometry, which are based on classical algebraic elimination algorithms (such as Gröbner bases computations) to propose a better-suited algebraic formulation.

Once such a formulation is obtained, the goal will be to adapt the algorithms of effective real algebraic geometry to the studied geometric situation. In particular, the aforementionned algorithms are based on computations of critical points (some special points which are local extrema of some polynomial mappings). In particular, exhibitting convexity and/or compactness properties can be exploited in the resolution process.

Required skills

Taste for both mathematics and programming. Enthusiasm is a must!

Possible continuation

The subject naturally leads to continuation in a PhD on the broader topic of nonlinear computational geometry using computer algebra.